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A branch-and-bound algorithm for maximizing the sum of several linear ratios

TAKAHITO KUNO
Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan (*E-mail*: *takahito@is.tsukuba.ac.jp*)

Abstract. In this paper, we develop a branch-and-bound algorithm for maximizing a sum of p (≥ 2) linear ratios on a polytope. The problem is embedded into a $2p$-dimensional space, in which a concave polyhedral function overestimating the optimal value is constructed for the bounding operation. The branching operation is carried out in a p-dimensional space, in a way similar to the usual rectangular branch-and-bound method. We discuss the convergence properties and report some computational results.

Key words: Global optimization; Nonconvex optimization; Fractional programming; Sum of linear ratios; Branch-and-bound algorithm

1. Introduction

Since the classical paper [4] by Charnes and Cooper in 1962, intensive research has been done on fractional programs [19]. Fractional programming is one of the most successful fields today in nonlinear optimization. In fact, the linear fractional program which optimizes a single linear ratio has been proved equivalent to a linear program by Charnes and Cooper [4]; and hence it can be solved in polynomial time now using interior-point algorithms [11]. Even in the multi-ratio case, the problem of maximizing the minimum value of linear ratios can be solved quite efficiently using a local search algorithm similar to Newton's method [7]. Unfortunately, however, there is still no decisive method for optimizing a sum of linear ratios on a polyhedron though it is also a multi-ratio problem.

The optimization of a sum of linear ratios arises in various areas: multi-stage stochastic shipping [1], cluster analysis [20] and multi-objective bond portfolio [15], to name but a few. While there is much demand for solution to this problem, all the theoretical results reported so far make us pessimistic about the existence of efficient algorithms [6, 18]. The only thing known about the optimality is that a globally optimal solution lies on the boundary of the feasible set if it exists [6]. During the last decade, however, some promising algorithms that use the low-rank nonconvexity [14] have been proposed for the problem with a few ratios [12, 16, 17]. As for the problem where the number of rarios is not limited, Falk and Palocsay [9] suggested an interesting approach in an 'image space'. They associated a new variable with each of the ratios and defined the image space, in which optimization is easy in

certain directions. Sequentially optimizing in these directions, they yielded a globablly optimal solution. Konno and Fukaishi [13] also associated a new variable with each of the ratios, thereby moving nonlinearities into the constraints. They further transformed the ratio constraints to multiplicative ones. To solve the resulting problem, they applied a branch-and-bound algorithm. In his recent paper [3], Benson solved nonlinear sum-of-ratios problems in a similar way; but he associated a new variable with only the reciprocal of each denominator. In a special case that the dimensionality of the problem is fixed at two, Chen et al. [5] recently developed a remarkably efficient algorithm using computational geometry.

In this paper, we will develop a branch-and-bound algorithm for maximizing a sum of p (≥ 2) linear ratios on a polytope. We associate a new variable with each of the denominators and numerators and define a $2p$-dimensional space. We construct a concave polyhedral function overestimating the value of the sum of ratios in this space and compute an upper bound on the optimal value. Therefore, the bounding operation is carried out mainly in the $2p$-dimensional space. In contrast to this, the stage of the branching operation is substantially the p-dimensional image space, i.e., we subdivide the range of each ratio successively in the algorithms. The organization of the paper is as follows. In Section 2, after giving the formal definition of the problem, we embed it into the $2p$-dimensional space. We also explain the outline of the branch-and-bound algorithm there. In Section 3, we construct the function overestimating the value of the sum of ratios. We then show that the lower bound can be computed by solving a linear program. Section 4 is devoted to the branching operation. We discuss the convergence of the algorithm and show that it is guaranteed if we subdivide the range of each ratio according to the same rules as adopted in the usual rectangular branch-and-bound method for separable concave minimization problems [10, 21]. In Section 5, we summarize the algorithm and prove that it generates a globally ϵ-optimally solution in finite time. Lastly, we report computational results in Section 6.

2. Reduction to $2p$-dimensional problem

Let us consider a problem of maximizing a sum of p linear ratios

$$\left|\begin{array}{ll} \text{maximize} & z = \displaystyle\sum_{i=1}^{p} \frac{\mathbf{d}^i\mathbf{x} + \delta_i}{\mathbf{c}^i\mathbf{x} + \gamma_i} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\,, \end{array}\right. \tag{2.1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c}^i$, $\mathbf{d}^i \in \mathbb{R}^n$ and γ_i, $\delta_i \in \mathbb{R}$ for $i = 1, \ldots, p$. We assume that the feasible set

$$X = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$$

is bounded and has a nonempty interior, and that

$$\mathbf{c}^i\mathbf{x} + \gamma_i > 0\,, \qquad \mathbf{d}^i\mathbf{x} + \delta_i > 0\,, \quad \forall \mathbf{x} \in X\,, \quad i = 1, \ldots, p\,. \tag{2.2}$$

As is well known, under condition (2.2) each ratio $(\mathrm{d}^i\mathbf{x} + \delta_i)/(\mathbf{c}^i\mathbf{x} + \gamma_i)$ is pseudomonotonic on X (i.e., both pseudoconvex and pseudoconcave; see [2] for details). The sum of pseudomonotonic functions is, however, neither psuedoconcave nor even quasiconcave in general. Therefore, (2.1) can have multiple locally optimal solutions, many of which fail to be globally optimal though at least one exists by compactness of X. What is even worse, no vertex of X might provide a globally optimal solution. This means that vertex enumeration often used in multiextremal global optimization [10] does not work on this problem (2.1).

For convenience, let us introduce two vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, each of p auxiliary variables, and define a $2p$-dimensional set:

$$\Omega = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{2p} \mid \boldsymbol{\xi} = \mathbf{C}\mathbf{x} + \gamma,\ \boldsymbol{\eta} = \mathbf{D}\mathbf{x} + \boldsymbol{\delta},\ \mathbf{x} \in X\},$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}^1 \\ \vdots \\ \mathbf{c}^p \end{bmatrix}, \qquad \gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \mathbf{d}^1 \\ \vdots \\ \mathbf{d}^p \end{bmatrix}, \qquad \boldsymbol{\delta} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_p \end{bmatrix}.$$

For each $i = 1, \ldots, p$, we also introduce four numbers s_i^1, t_i^1, u_i and v_i satisfying

$$\left.\begin{aligned} &0 < s_i^1 \le \min\{(\mathbf{d}^i\mathbf{x} + \delta_i)/(\mathbf{c}^i\mathbf{x} + \gamma_i) \mid \mathbf{x} \in X\} \\ &\infty > t_i^1 \ge \max\{(\mathbf{d}^i\mathbf{x} + \delta_i)/(\mathbf{c}^i\mathbf{x} + \gamma_i) \mid \mathbf{x} \in X\} \\ &0 < u_i \le \min\{(\mathbf{c}^i + \mathbf{d}^i)\mathbf{x} \mid \mathbf{x} \in X\} + \gamma_i + \delta_i \\ &\infty > v_i \le \max\{(\mathbf{c}^i + \mathbf{d}^i)\mathbf{x} \mid \mathbf{x} \in X\} + \gamma_i + \delta_i. \end{aligned}\right\} \tag{2.3}$$

Notice that we can easily obtain each of these numbers by solving a linear programming problem [4]. Let

$$\Gamma_i = \{(\xi_i, \eta_i) \in \mathbb{R}_+^2 \mid u_i \le \xi_i + \eta_i \le v_i\}$$
$$\Delta_i^1 = \{(\xi_i, \eta_i) \in \mathbb{R}_+^2 \mid s_i^1 \xi_i \le \eta_i \le t_i^1 \xi_i\},$$

where $\mathbb{R}_+^{\cdot}$ denotes the nonnegative orthant of $\mathbb{R}^{\cdot}$; and let

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_p, \qquad \Delta^1 = \Delta_1^1 \times \cdots \times \Delta_p^1.$$

Since Ω is a subset of $\Gamma \cap \Delta^1$, problem (2.1) reduces to a $2p$-dimensional master problem

$$\mathrm{MP} \left| \begin{array}{ll} \text{maximize} & z = \displaystyle\sum_{i=1}^{p} \eta_i/\xi_i \\ \text{subject to} & (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \Omega \cap \Gamma \cap \Delta^1. \end{array} \right.$$

We apply a branch-and-bound method to this problem, instead of the original problem (2.1) of dimensionality n.

In our algorithm, while partitioning the cone Δ^1 successively into

$$\Delta^j = \Delta_1^j \times \cdots \times \Delta_p^j, \quad j \in \mathcal{J}, \tag{2.4}$$

we solve each subproblem of MP with a feasible set $\Omega \cap \Gamma \cap \Delta^j$, where

$$\left.\begin{aligned} &\Delta_i^j = \{(\xi_i, \eta_i) \in \mathbb{R}_+^2 \mid s_i^j \xi_i \le \eta_i \le t_i^j \xi_j\} \\ &\Delta_i^1 = \cup_{j \in \mathscr{J}} \Delta_i^j \,;\ \text{int}\, \Delta_i^j \cap \text{int}\, \Delta_i^k = \emptyset \ \text{if}\ j \ne k\,. \end{aligned}\right\} \tag{2.5}$$

The outline is as follows:

Let $\mathscr{J} := \{1\}$ and $k := 1$. Repeat Steps 1–3 while $\mathscr{J} \ne \emptyset$.

Step 1. Take an appropriate index j from $\mathscr{J}$ and let $\Delta := \Delta^j$. Define a subproblem

$$\text{P}(\Delta) \left| \begin{array}{ll} \text{maximize} & z = \displaystyle\sum_{i=1}^{p} \eta_i / \xi_i \\ \text{subject to} & (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \Omega \cap \Gamma \cap \Delta\,. \end{array} \right.$$

Step 2 (bounding operation). Compute an upper bound $\bar{z}(\Delta)$ on the value of P(Δ). If $\bar{z}(\Delta)$ is less than or equal to the value of the best feasible solution obtained so far, discard Δ and return to Step 1.

Step 3 (branching operation). Otherwise, divide Δ into two cones Δ^{2k} and Δ^{2k+1}. Add $\{2k, 2k+1\}$ to $\mathscr{J}$ and $k := k+1$.

Needless to say, the efficiency of this algorithm is most influenced by Steps 2 and 3. We will show how to carry them out in order. Throughout the paper, we identify $\mathscr{J}$ with the set of cones Δ^j, $j \in \mathscr{J}$.

3. Bounding operation (Step 2)

The cone Δ defining problem P(Δ) is a direct product of p cones, each in a two-dimensional plane:

$$\Delta_i = \{(\xi_i, \eta_i) \in \mathbb{R}_+^2 \mid s_i \xi_i \le \eta_i \le t_i \xi_i\}, \quad i = 1, \ldots, p\,.$$

Therefore, $\Gamma_i \cap \Delta_i$ for each i constitutes a trapezoid with four vertices:

$$S = (u_i, s_i u_i)/(s_i + 1)\,, \qquad T = (v_i, s_i v_i)/(s_i + 1)$$
$$U = (v_i, t_i v_i)/(t_i + 1)\,, \qquad V = (u_i, t_i u_i)/(t_i + 1)$$

(see Figure 1). Let

$$\left.\begin{aligned} &f_i(\xi_i, \eta_i) = (t_i + 1)(\eta_i - s_i \xi_i)/u_i + s_i \\ &g_i(\xi_i, \eta_i) = (s_i + 1)(\eta_i - t_i \xi_i)/v_i + t_i\,. \end{aligned}\right\} \tag{3.1}$$

As is shown in Figure 2, f_i is an affine function passing η_i/ξ_i at vertex V and side

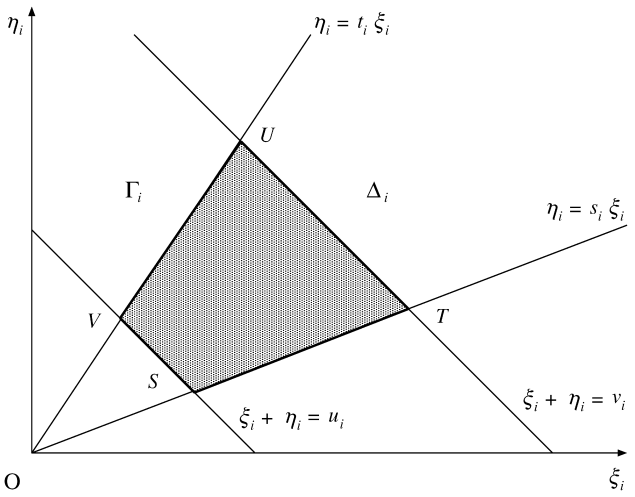

Figure 1. Trapezoid $\Gamma_i \cap \Delta_i$.

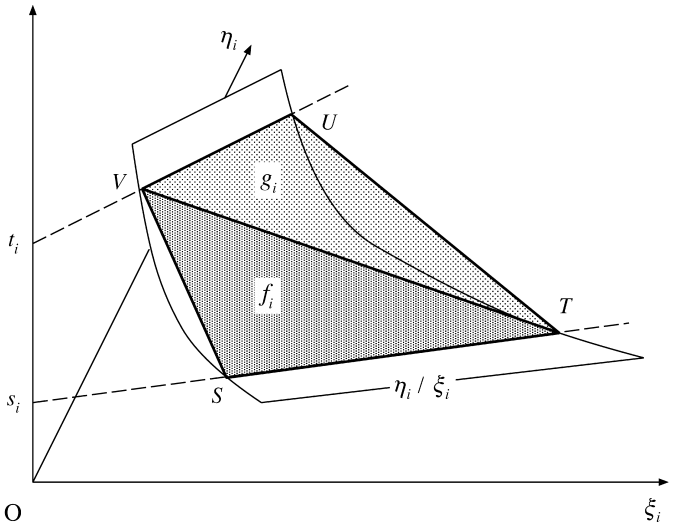

Figure 2. Overestimator ϕ_i of η_i/ξ_i.

S–T of the trapezoid; and g_i passes it at vertex T and side U–V. Using these two functions, we define

$$\phi_i(\xi_i, \eta_i) = \min\{f_i(\xi_i, \eta_i), g_i(\xi_i, \eta_i)\}\,. \tag{3.2}$$

LEMMA 3.1. *The function ϕ_i is concave, polyhedral and satisfies the following for any $(\xi_i, \eta_i) \in \Gamma_i$:*

$$\left.\begin{aligned} &\phi_i(\xi_i, \eta_i) \geq \eta_i/\xi_i \text{ if } (\xi_i, \eta_i) \in \Delta_i \\ &\phi_i(\xi_i, \eta_i) < \eta_i/\xi_i \text{ if } (\xi_i, \eta_i) \notin \Delta_i\,. \end{aligned}\right\} \tag{3.3}$$

In particular, the value of ϕ_i agrees with η_i/ξ_i at two sides S–T $(\eta_i/\xi_i = s_i)$ and U-V $(\eta_i/\xi_i = t_i)$ of trapezoid $\Gamma_i \cap \Delta_i$.

Proof. Let us divide $\Gamma_i = \{(\xi_i, \eta_i) \,|\, u_i \leq \xi_i + \eta_i \leq v_i\}$ by line T–V into

$$\Gamma_i^{\mathrm{f}} = \Gamma_i \cap \{(\xi_i, \eta_i) \,|\, f_i(\xi_i, \eta_i) \leq g_i(\xi_i, \eta_i)\}$$
$$\Gamma_i^{g} = \Gamma_i \cap \{(\xi_i, \eta_i) \,|\, f_i(\xi_i, \eta_i) \geq g_i(\xi_i, \eta_i)\}$$

(see Figure 2), and take an arbitrary point (ξ_i', η_i') from Γ_i^f. We then have $\phi_i(\xi_i', \eta_i') = f_i(\xi_i', \eta_i')$. If $\eta_i' = 0$, then (ξ_i', η_i') cannot be a point in Δ_i; and hence the second of (3.3) holds. Assuming $\eta_i' \neq 0$, we have

$$\begin{aligned} \phi_i(\xi_i', \eta_i') - \eta_i'/\xi_i' &= (t_i + 1)(\eta_i' - s_i\xi_i')/u_i - (\eta_i' - s_i\xi_i')/\xi_i' \\ &= (\eta_i' - s_i\xi_i')(t_i\xi_i' + \xi_i' - u_i)/(u_i\xi_i')\,. \end{aligned}$$

If $(\xi_i', \eta_i') \in \Delta_i$, then $s_i\xi_i' \leq \eta_i' \leq t_i\xi_i'$ and

$$\phi_i(\xi_i', \eta_i') - \eta_i'/\xi_i' \geq (\eta_i' - s_i\xi_i')(t_i\xi_i' - \eta_i')/(u_i\xi_i') \geq 0\,.$$

Otherwise, $\eta_i' < s_i\xi_i' \leq t_i\xi_i'$ and

$$\phi_i(\xi_i', \eta_i') - \eta_i'/\xi_i' \leq (\eta_i' - s_i\xi_i')(t_i\xi_i' - \eta_i')/(u_i\xi_i') < 0\,.$$

Similarly, (3.3) holds for any point in Γ_i^g. The rest follows from definition. □

We refer to ϕ_i as the *overestimator* of η_i/ξ_i on $\Gamma_i \cap \Delta_i$. Replacing η_i/ξ_i by its overestimator for each i in P(Δ), we have a concave maximization problem

$$\bar{\mathrm{P}}(\Delta) \left|\begin{array}{ll} \text{maximize} & z = \displaystyle\sum_{i=1}^{p} \phi_i(\xi_i, \eta_i) \\ \text{subject to} & (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \Omega \cap \Gamma \cap \Delta\,. \end{array}\right.$$

Let $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$ be an optimal solution to $\bar{\mathrm{P}}(\Delta)$ and $\bar{z}(\Delta)$ the value of $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$. Also let $z(\Delta)$ denote the optimal value of P(Δ). If $\Omega \cap \Gamma \cap \Delta = \emptyset$, we interpret both $z(\Delta)$ and $\bar{z}(\Delta)$ as $-\infty$. The following is an immediate consequence of Lemma 3.1.

LEMMA 3.2. *If* $\bar{z}(\Delta) > -\infty$, *then*

$$\sum_{i=1}^{p} \bar{\eta}_i / \bar{\xi}_i \le z(\Delta) \le \bar{z}(\Delta) . \tag{3.4}$$

3.1. SOLUTION TO $\bar{\mathrm{P}}(\Delta)$

We should remark that $\bar{\mathrm{P}}(\Delta)$ is equivalent to a linear programming problem

$$\left|\begin{array}{ll} \text{maximize} & z = \sum_{i=1}^{p} \zeta_i \\ \text{subject to} & \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \ge \mathbf{0} \\ & \left.\begin{array}{l} (\mathrm{t_i}+1)(\mathbf{d}^i - s_i\mathbf{c}^i)\mathbf{x} - u_i\zeta_i \ge \alpha_i \\ (s_i+1)(\mathbf{d}^i - t_i\mathbf{c}^i)\mathbf{x} - v_i\zeta_i \ge \beta_i \\ s_i \le \zeta_i \le t_i \end{array}\right\} i = 1, \ldots, p , \end{array}\right. \tag{3.5}$$

where

$$\alpha_i = (t_i+1)(s_i\gamma_i - \delta_i) - s_iu_i , \qquad \beta_i = (s_i+1)(t_i\gamma_i - \delta_i) - t_iv_i .$$

To see this, we need first to prove the following:

LEMMA 3.3. *Let* (ξ_i, η_i) *be a point in* Γ_i. *Then*

$$(\xi_i, \eta_i) \in \Delta_i \text{ iff } s_i \le \phi_i(\xi_i, \eta_i) \le t_i . \tag{3.6}$$

Proof. Let Γ_i^f and Γ_i^g denote the subsets of Γ_i defined in the proof of Lemma 3.1. Then $\Gamma_i^f \cap \Delta_i$ is a triangle with vertices S, T and V (see Figure 2). It is easy to check that side S–T and vertex V provide the minimum value s_i and the maximum value t_i, respectively, for $\phi_i = f_i$ on $\Gamma_i^f \cap \Delta_i$. If $(\xi_i, \eta_i) \in \Gamma_i^f \backslash \Delta_i$, then $\eta_i < s_i\xi_i$ and hence $\phi_i(\xi_i, \eta_i) = (t_i+1)(\eta_i - s_i\xi_i)/u_i + s_i < s_i$. Therefore, (3.6) holds for any $(\xi_i, \eta_i) \in \Gamma_i^f$. Similarly, we can show (3.6) for $(\xi_i, \eta_i) \in \Gamma_i^g$. □

PROPOSITION 3.4. *If (3.5) is infeasible, then* $\bar{z}(\Delta) = -\infty$. *Otherwise, for any optimal solution* $(\bar{\mathbf{x}}, \bar{\boldsymbol{\xi}})$ *to (3.5) we have*

$$\bar{\boldsymbol{\xi}} = \mathbf{C}\bar{\mathbf{x}} + \boldsymbol{\gamma} , \qquad \bar{\boldsymbol{\eta}} = \mathbf{D}\bar{\mathbf{x}} + \boldsymbol{\delta} , \qquad \bar{z}(\Delta) = \sum_{i=1}^{p} \bar{\zeta}_i .$$

Proof. We see from (3.6) that $\bar{\mathrm{P}}(\Delta)$ is equivalent to

$$\left|\begin{array}{ll} \text{maximize} & z = \sum_{i=1}^{p} \zeta_i \\ \text{subject to} & (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \Omega \\ & \left.\begin{array}{l} \zeta_i = \phi_i(\xi_i, \eta_i) \\ s_i \le \zeta_i \le t_i \end{array}\right\} i = 1, \ldots, p . \end{array}\right.$$

This problem reduces to (3.5); and hence the assertion follows. □

Problem $\bar{P}(\Delta)$ itself is actually a linear programming problem even if we do not replace the constraint $(\xi_i, \eta_i) \in \Delta_i$ by $s_i \le \phi_i(\xi_i, \eta_i) \le t_i$ for each i. This transformation, however, enables one to compute $\bar{z}(\Delta)$ using the upper bounding simplex method and to reduce the total computational time considerably (see [8] for details). Anyway, by solving a linear programming problem, we obtain an upper bound $\bar{z}(\Delta)$ on the value of $P(\Delta)$ and its feasible solution $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$, both needed in Step 2 of the algorithm.

4. Branching operation (Step 3)

In Step 3, we have to divide Δ in such a way that the resulting sets Δ^{2k} and Δ^{2k+1} satisfy (2.4) and (2.5). This can be done by giving an index $i \in \{1, \ldots, p\}$ and a number $w_i \in [s_i, t_i]$. Namely,

$$\Delta^j = \Delta_1 \times \cdots \times \Delta_{i-1} \times \Delta_i^j \times \Delta_{i+1} \times \cdots \times \Delta_p\,, \quad j = 2k, 2k+1\,, \tag{4.1}$$

where

$$\left.\begin{aligned} \Delta_i^{2k} &= \{(\xi_i, \eta_i) \in \mathbb{R}_+^2 \mid s_i \xi_i \le \eta_i \le w_i \xi_i\} \\ \Delta_i^{2k+1} &= \{(\xi_i, \eta_i) \in \mathbb{R}_+^2 \mid w_i \xi_i \le \eta_i \le t_i \xi_i\}\,, \end{aligned}\right\} \tag{4.2}$$

In general, no matter how we select i and w_i, the finiteness of the algorithm cannot be guaranteed without a tolerance for the optimal value of problem (2.1). In that case, the algorithm generates an infinite sequence of cones Δ^{j_ℓ}, $\ell = 1, 2, \ldots$ such that

$$\Delta^{j_1} \supset \Delta^{j_2} \supset \cdots, \qquad \Omega \cap \Gamma \cap \left(\bigcap_{\ell=1}^{\infty} \Delta^{j_\ell} \right) \neq \emptyset\,. \tag{4.3}$$

Let us denote Δ^{j_ℓ} simply by Δ^ℓ and the sequence by the index set $\mathcal{L} = \{1, 2, \ldots, \ell, \ldots\}$. We assume that for each $\ell \in \mathcal{L}$, cone $\Delta^{\ell+1}$ is generated from Δ^ℓ via (4.1) and (4.2) for some $i_\ell \in \{1, \ldots, p\}$ and $w_{i_\ell}^\ell \in [s_{i_\ell}^\ell, t_{i_\ell}^\ell]$, where $\Delta_{i_\ell}^\ell = \{(\xi_{i_\ell}, \eta_{i_\ell}) \in \mathbb{R}_+^2 \mid s_{i_\ell}^\ell \eta_{i_\ell} \le \xi_{i_\ell} \le t_{i_\ell}^\ell \eta_{i_\ell}\}$. The following lemma shows that $\mathcal{L}$ possesses a property similar to nested rectangles generated by the rectangular branch-and-bound method for separable concave minimization problems (see Lemma 5.4 in [21]):

LEMMA 4.1. *There exists an infinite subsequence $\mathcal{L}_q \subset \mathcal{L}$ such that $i_\ell = q$ for all $\ell \in \mathcal{L}_q$. Also, $\{s_q^\ell \mid \ell \in \mathcal{L}_q\}$ and $\{t_q^\ell \mid \ell \in \mathcal{L}_q\}$ have limits s_q^* and t_q^* such that $s_q^* \le t_q^*$; and $\{w_q^\ell \mid \ell \in \mathcal{L}_q\}$ converges to $w_q^* \in \{s_q^*, t_q^*\}$.*

Proof. Since i_ℓ is an element of the finite set $\{1, \ldots, p\}$, we can take an infinite subsequence $\mathcal{L}_q$ such that $i_\ell = q$ for all $\ell \in \mathcal{L}_q$. Assuming $\mathcal{L}_q = \{1, 2, \ldots\}$ without loss generality, we have

$$s_q^1 \le s_q^\ell \le s_q^{\ell+1} \le t_q^{\ell+1} \le t_q^\ell \le t_q^1\,, \quad \forall\, \ell \in \mathcal{L}_q\,.$$

Hence, for some s_q^* and t_q^* such that $s_q^1 \le s_q^* \le t_q^* \le t_q^1$ we have

$$\lim_{\ell\to\infty} s_q^\ell = \lim_{\ell\to\infty} s_q^{\ell+1} = s_q^*\,, \qquad \lim_{\ell\to\infty} t_q^\ell = \lim_{\ell\to\infty} t_q^{\ell+1} = t_q^*\,.$$

These also imply that $\lim_{\ell\to\infty} w_q^\ell = w_q^* \in \{s_q^*, t_q^*\}$ because w_q^ℓ coincides with either $s_q^{\ell+1}$ or $t_q^{\ell+1}$ for each $\ell \in \mathscr{L}_q$. □

In the rest of this section, we will give two different rules of selecting $(i_\ell, w_{i_\ell}^\ell)$ to divide Δ^ℓ for each $\ell \in \mathscr{L}$. The sequence $\mathscr{L}$ generated by each of these rules satisfies

$$\lim_{\ell\to\infty} \left[\bar{z}(\Delta^\ell) - \sum_{i=1}^{p} \bar{\eta}_i^\ell / \bar{\xi}_i^\ell \right] = 0\,, \tag{4.4}$$

where $(\bar{\boldsymbol{\xi}}^\ell, \bar{\boldsymbol{\eta}}^\ell)$ is an optimal solution to $\bar{\mathrm{P}}(\Delta^\ell)$ and $\bar{z}(\Delta^\ell)$ the optimal value. The condition (4.4) is the key to guarantee the finiteness of the algorithm when a positive tolerance is allowed for the optimal value of problem (2.1).

4.1. BISECTION

On the analogy of the rectangular branch-and-bound method, the easiest way to divide Δ^ℓ is bisection. For each $\ell \in \mathscr{L}$, let us select

$$i_\ell \in \arg\max\{t_i^\ell - s_i^\ell \mid i = 1, \ldots, p\}\,; \tag{4.5}$$

and divide $\Delta_{i_\ell}^\ell$ by the line $\eta_{i_\ell} = w_{i_\ell}^\ell$ for

$$w_{i_\ell}^\ell = (1-\lambda)s_{i_\ell}^\ell + \lambda t_{i_\ell}^\ell\,, \tag{4.6}$$

where $\lambda \in (0, 1)$ is a constant. We refer to this selection rule of $(i_\ell, w_{i_\ell}^\ell)$ as *bisection* of ratio λ.

LEMMA 4.2. *If $\mathscr{L}$ is generated according to the bisection rule of ratio $\lambda \in (0, 1)$, then (4.4) holds.*

Proof. As in Lemma 4.1, let $\mathscr{L}_q \subset \mathscr{L}$ denote the infinite sequence where $i_\ell = q$ for all ℓ. Then we have $s_q^\ell \to s_q^*$, $t_q^\ell \to t_q^*$ and $w_q^\ell \to w_q^* \in \{s_q^*, t_q^*\}$ as $\ell \to \infty$ in $\mathscr{L}_q$. From (4.6), however, we have

$$(1-\lambda)s_q^* + \lambda t_q^* = w_q^* \in \{s_q^*, t_q^*\}\,,$$

which holds only if $w_q^* = s_q^* = t_q^*$. This, together with (4.5), implies that if $k \to \infty$ in $\mathscr{L}$, cone Δ^ℓ shrinks to a half-line:

$$\Delta^* = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}_+^{2p} \mid \xi_i = w_i^* \eta_i,\ i = 1, \ldots, p\}\,, \tag{4.7}$$

where w_i^* is some point in $[s_i^1, t_i^1]$.

For each i, the sequence $\{(\xi_i^\ell, \bar{\eta}_i^\ell) \mid \ell \in \mathscr{L}\}$ is generated in the compact set $\Gamma_i \cap \Delta_i^1$, and hence has at least one limit point (ξ_i^*, η_i^*), which satisfies $\xi_i^* = w_i^* \eta_i^*$ by (4.7). Therefore,

$$\lim_{\ell\to\infty} \bar{\eta}_i^{\ell}/\bar{\xi}_i^{\ell} = w_i^* .$$

On the other hand, we have $s_i^{\ell} \le \phi_i^{\ell}(\bar{\xi}_i^{\ell}, \bar{\eta}_i^{\ell}) \le t_i^{\ell}$ from Lemma 3.3, where ϕ_i^{ℓ} denotes the overestimator of η_i/ξ_i in $\Gamma_i \cap \Delta_i^{\ell}$. Hence,

$$\lim_{\ell\to\infty} \phi_i^{\ell}(\bar{\xi}_i^{\ell}, \bar{\eta}_i^{\ell}) = w_i^* . \tag{4.9}$$

Since $\bar{z}(\Delta^{\ell}) = \Sigma_{i=1}^{p} \phi_i^{\ell}(\bar{\xi}_i^{\ell}, \bar{\boldsymbol{\eta}}_i^{\ell})$, the condition (4.4) follows from (4.8) and (4.9). □

4.2. ω-DIVISION

The bisection rule is simple but does not entirely exploit the characteristics of problem $\bar{P}(\Delta^{\ell})$. As stated in Lemma 3.1, the overestimator ϕ_i^{ℓ} composing the objective function agrees with η_i/ξ_i on two sides of trapezoid $\Gamma_i \cap \Delta_i^{\ell}$. The next selection rule of $(i_{\ell}, w_{i_{\ell}}^{\ell})$ uses this property of ϕ_i^{ℓ} to fulfill the condition (4.4).

For each $\ell \in \mathcal{L}$, let us select

$$i_{\ell} \in \arg\max\{\phi_i^{\ell}(\bar{\xi}_i^{\ell}, \bar{\eta}_i^{\ell}) - \bar{\eta}_i^{\ell}/\bar{\xi}_i^{\ell} \mid i = 1, \ldots, p\}; \tag{4.10}$$

and let

$$w_{i_{\ell}}^{\ell} = \bar{\eta}_{i_{\ell}}^{\ell}/\bar{\xi}_{i_{\ell}}^{\ell} . \tag{4.11}$$

This kind of selection rules is often called *ω-division* in global optimization branch-and-bound methods (see [10]); and we follow the custom.

LEMMA 4.3. *If $\mathcal{L}$ is generated according to the ω-division rule, then (4.4) holds.*

Proof. Suppose that $w_q^{\ell} \to w_q^* = s_q^*$ as $k \to \infty$ in $\mathcal{L}_q \subset \mathcal{L}$, where $\mathcal{L}_q$ is an infinite sequence with $i_{\ell} = q$ for all $l \in \mathcal{L}_q$. Let $\mathcal{L}'_q$ be a subsequence of $\mathcal{L}_q$ such that $w_q^{\ell} = s_q^{\ell+1}$ for all $\ell \in \mathcal{L}'_q$. Then we have $\bar{\eta}_q^{\ell}/\bar{\xi}_q^{\ell} = s_q^{\ell+1}$ from (4.11), and $\phi_q^{\ell+1}(\bar{\xi}_q^{\ell}, \bar{\eta}_q^{\ell}) = s_q^{\ell+1}$ from Lemma 3.1. If $\ell \to \infty$ in $\mathcal{L}'_q$, then $\bar{\eta}_q^{\ell}/\bar{\xi}_q^{\ell} \to s_q^*$ and $\phi_q^{\ell+1}(\bar{\xi}_q^{\ell}, \bar{\eta}_q^{\ell}) \to s_q^*$. Hence, we have

$$\lim_{\ell\to\infty} [\phi_q^{\ell}(\bar{\xi}_q^{\ell}, \bar{\eta}_q^{\ell}) - \bar{\eta}_q^{\ell}/\bar{\xi}_q^{\ell}] = 0 . \tag{4.12}$$

Even when $w_q^* = t_q^*$, we have the same result. The condition (4.4) follows from (4.12). □

5. Description of the algorithm

The last thing to be discussed is how to select a cone Δ^j from the set $\mathcal{J}$ in Step 1. In the usual branch-and-bound methods, either of the following rules is adopted:

Depth first. The set $\mathcal{J}$ is maintained as a list of *stack*. A cone Δ^j is taken from the top of $\mathcal{J}$; and cones Δ^{2k+1} and Δ^{2k} are added in this order to the top.

Best bound. The set $\mathcal{J}$ is maintained as a list of *priority queue*. A cone Δ^j of largest $\bar{z}(\Delta^j)$ is taken out of $\mathcal{J}$.

We can naturally use either in our algorithm. In addition to this, if we incorporate the bisection or ω-division rule into Step 2, our algorithm is completed.

Let $\epsilon \geq 0$ be a given tolerance for the optimal value of problem (2.1). Then the algorithm is summarized as follows:

algorithm SUMRATIO.
begin
  for $i = 1, \ldots, p$ **do begin**
    compute s_i^1, t_i^1, u_i and v_i;
    $\Gamma_i := \{(\xi_i, \eta_i) \in \mathbb{R}^2_+ \mid u_i \leq \xi_i + \eta_i \leq v_i\}$; $\Delta_i := \{(\xi_i, \eta_i) \in \mathbb{R}^2_+ \mid s_i^1 \xi_i \leq \eta_i \leq t_i^1 \xi_i\}$
  end;
  $\Gamma := \Gamma_1 \times \cdots \times \Gamma_p$; $\Delta^1 := \Delta_1^1 \times \cdots \times \Delta_p^1$; $\mathcal{J} := \{1\}$; $z^\epsilon := 0$; $k := 1$;
  while $\mathcal{J} \neq \emptyset$ **do begin**
    select $j \in \mathcal{T}$ by a fixed rule (*depth first or best bound*); /* *Step 1* */
    $\mathcal{J} := \mathcal{J} \backslash \{j\}$; set $\Delta := \Delta^j$ and define a subproblem P(Δ);
    for $i = 1, \ldots, p$ **do** /* *Step 2* */
      determine the overestimator ϕ_i of η_i/ξ_i on $\underline{\Gamma}_i \cap \Delta_i$;
    construct the concave maximization problem $\bar{\mathrm{P}}(\Delta)$ using ϕ_i's;
    solve $\bar{\mathrm{P}}(\Delta)$ to obtain an uper bound $\bar{z}(\Delta)$ on the value of P(Δ);
    if $\bar{z}(\Delta) - z^\epsilon > \epsilon$ **then begin** /* *Step 3* */
      let $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$ be a solution of value $\bar{z}(\Delta)$ to $\bar{\mathrm{P}}(\Delta)$;
      if $\sum_{i=1}^p \bar{\eta}_i/\bar{\xi}_i > z^\epsilon$ **then**
        update $z^\epsilon := \sum_{i=1}^p \bar{\eta}_i/\bar{\xi}_i$ and $(\boldsymbol{\xi}^\epsilon, \boldsymbol{\eta}^\epsilon) := (\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$;
      select $i \in \{1, \ldots, p\}$ and $w_i \in [s_i, t_i]$ by a fixed rule (*bisection or ω-division*);
      $\Delta_i^{2k} := \{(\xi_i, \eta_i) \in \mathbb{R}^2_+ \mid s_i \xi_i \leq \eta_i \leq w_i \xi_i\}$;
      $\Delta_i^{2k+1} := \{(\xi_i, \eta_i) \in \mathbb{R}^2_+ \mid w_i \xi_i \leq \eta_i \leq t_i \xi_i\}$;
      $\Delta^j := \Delta_1 \times \cdots \Delta_{i-1} \times \Delta_i^j \times \Delta_{i+1} \times \cdots \times \Delta_p$ for $j = 2k,\ 2k+1$;
      $\mathcal{J} := \mathcal{J} \cup \{2k, 2k+1\}$; $k := k + 1$
    end
  end;
  let $\mathbf{x}^e$ be a feasible solution to (2.1) such that $\boldsymbol{\xi}^\epsilon = \mathbf{C}\mathbf{x}^\epsilon + \boldsymbol{\gamma}$ and $\boldsymbol{\eta}^\epsilon = \mathbf{D}\mathbf{x}^\epsilon + \boldsymbol{\delta}$
end;

THEOREM 5.1. *When $\epsilon > 0$, the algorithm SUMRATIO terminates in a finite number of iterations and yields a globally ϵ-optimal solution $\mathbf{x}^\epsilon$ to problem (2.1).*

Proof. Let us assume the contrary: the algorithm SUMRATIO is infinite. Then it generates an infinite sequence $\mathcal{L}$ of Δ^{j_i}'s satisfying (4.3). The backtracking criterion

$\bar{z}(\Delta) - z^{\epsilon} > \epsilon$ implies that the following inequalities hold at the end of each iteration in which j^{ℓ} for $\ell \in \mathscr{L}$ is taken out of $\mathscr{T}$:

$$\sum_{i=1}^{p} \bar{\eta}_i^{\ell} / \bar{\xi}_i^{\ell} \leq z^{\epsilon} < \bar{z}(\Delta^{\ell}) - \epsilon ,$$

where $\Delta^{\ell} = \Delta^{j_\ell}$. From Lemmas 4.2 and 4.3, however, $\lim_{\ell \to \infty} \bar{z}(\Delta^{\ell}) - \Sigma_{i=1}^{p} \bar{\eta}_i^{\ell} / \bar{\xi}_i^{\ell} = 0$ whichever rule we adopt for selecting (i, w_i) in Step 3. Therefore, $\epsilon \leq 0$, which is a contradiction. The ϵ-optimality of $\mathbf{x}^{\epsilon}$ follows from the backtracking criterion. □

COROLLARY 5.2. *Suppose* $\epsilon = 0$. *If the best bound rule is adopted in Step 1, the sequence of* $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$*'s generated by the algorithm SUMRATIO has limit points, each of which is a globally optimal solution to problem MP.*

Proof. If the algorithm happens to be finite, the assertion is obvious from the backtracking criterion. Assume that it is infinite and generates an infinite sequence $\mathscr{L}$ just stated in the proof of the previous theorem. The best bound rule then implies the following at the beginning of each iteration in which j_ℓ for $\ell \in \mathscr{L}$ is taken out of $\mathscr{T}$:

$$\bar{z}(\Delta^{\ell}) \geq \bar{z}(\Delta^{j}) \geq z(\Delta^{j}) , \quad \forall j \in \mathscr{J} ,$$

where $\Delta^{\ell} = \Delta^{j_\ell}$. However, $\lim_{\ell \to \infty} \bar{z}(\Delta^{\ell}) - \Sigma_{i=1}^{p} \bar{\eta}_i^{\ell} / \bar{\xi}_i^{\ell} = 0$; and besides $\max\{z(\Delta^{j}) \mid j \in \mathscr{J}\}$ is nothing but the optimal value of MP. Hence, every limit point $\{(\bar{\boldsymbol{\xi}}^{\ell}, \bar{\boldsymbol{\eta}}^{\ell}) \ \ell \in \mathscr{L}\}$ is a globally optimal solution to MP. □

5.1. NUMERICAL EXAMPLES

To illustrate the algorithm SUMRATIO, let us compute a globally 10^{-3}-optimal solution to the following small instance according to the depth-first and ω-division rules:

$$\begin{array}{ll} \text{maximize} & z = \dfrac{3x_1 + 5x_2 + 3x_3 + 50}{3x_1 + 4x_2 + 5x_3 + 50} + \dfrac{3x_1 + 4x_2 + 50}{4x_1 + 3x_2 + 2x_3 + 50} + \\ & \dfrac{4x_1 + 2x_2 + 4x_3 + 50}{5x_1 + 4x_2 + 3x_3 + 50} \\ \text{subject to} & 6x_1 + 3x_2 + 3x_3 \leq 10,\ 10x_1 + 3x_2 + 8x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 . \end{array} \tag{5.1}$$

Before starting iteration, we have to determine the numbers s_i^1, t_i^1, u_i and v_i for $i = 1, 2, 3$. Although these numbers are not specified in the description of the algorithm, only one linear program yields them for each i in this particular case. namely, v_i is determined as the maximum value of $\Sigma_{j=1}^{3} (c_{ij} + d_{ij}) x_j + 100$ by solving a linear programming problem with the same constraints as (5.1). As for u_i, it can be set to 100 because both $\Sigma_{j=1}^{3} c_{ij} x_j + 50$ and $\Sigma_{j=1}^{3} d_{ij} x_j + 50$ have the

minimum value 50. Then, s_i^1 and t_i^1 are given by $50/(v_i - 50)$ and $(v_i - 50)/50$, respectively. It is easily seen that these s_i^1, t_i^1, u_i and v_i satisfy (2.3), though the resulting $\Gamma \cap \Delta^1$ is somewhat baggy to wrap up Ω:

$$\Gamma_1 = \{(\xi_1, \eta_1) \in \mathbb{R}^2_+ \mid 100.000 \le \xi_1 + \eta_1 \le 130.000\}$$
$$\Gamma_2 = \{(\xi_2, \eta_2) \in \mathbb{R}^2_+ \mid 100.000 \le \xi_2 + \eta_2 \le 123.333\}$$
$$\Gamma_3 = \{(\xi_3, \eta_3) \in \mathbb{R}^2_+ \mid 100.000 \le \xi_3 + \eta_3 \le 120.000\}$$
$$\Delta_1^1 = \{(\xi_1, \eta_1) \in \mathbb{R}^2_+ \mid 0.625\xi_1 \le \eta_1 \le 1.600\xi_1\}$$
$$\Delta_2^1 = \{(\xi_2, \eta_2) \in \mathbb{R}^2_+ \mid 0.682\xi_2 \le \eta_2 \le 1.467\xi_2\}$$
$$\Delta_3^1 = \{(\xi_3, \eta_3) \in \mathbb{R}^2_+ \mid 0.714\xi_1 \le \eta_3 \le 1.400\xi_3\}\,.$$

As a by-product of this preprocess, we have at least one feasible solution to (5.1), which can be used as the initial incumbent:

$$\mathbf{x}^\epsilon = (0.000,\ 3.333,\ 0.000)\,, \qquad z^\epsilon = 3.015$$
$$(\boldsymbol{\xi}^\epsilon;\ \boldsymbol{\eta}^\epsilon) = (66.667,\ 63.333,\ 56.667;\ 63.333,\ 60.000,\ 63.333)\,.$$

After storing $\Delta^1 = \Delta_1^1 \times \Delta_2^1 \times \Delta_3^1$ in $\mathcal{J}$, we proceed to the iterative process.

Iteration 1. We select Δ^1 as Δ from $\mathcal{J}$ and let $\mathcal{J} = \mathcal{J}\backslash\{1\}$. According to (3.1) and (3.2), we determine the overestimator of ξ_i/η_i on $\Gamma_i \cap \Delta_i$ for $i = 1, 2, 3$:

$$\phi_1(\xi_1, \eta_1) = \min\begin{Bmatrix} -0.016\xi_1 + 0.026\eta_1 + 0.625 \\ -0.020\xi_1 + 0.013\eta_1 + 1.600 \end{Bmatrix}$$
$$\phi_2(\xi_2, \eta_2) = \min\begin{Bmatrix} -0.017\xi_2 + 0.025\eta_2 + 0.682 \\ -0.020\xi_2 + 0.014\eta_2 + 1.467 \end{Bmatrix}$$
$$\phi_3(\xi_3, \eta_3) = \min\begin{Bmatrix} -0.017\xi_3 + 0.024\eta_3 + 0.714 \\ -0.020\xi_3 + 0.014\eta_3 + 1.400 \end{Bmatrix}.$$

The problem $\bar{P}(\Delta)$ of maximizing the sum of these three concave functions is equivalent to a linear programming problem:

$$\begin{array}{ll} \text{maximize} & z = \zeta_1 + \zeta_2 + \zeta_3 \\ \text{subject to} & 6x_1 + 3x_2 + 3x_3 \le 10,\ 10x_1 + 3x_2 + 8x_3 \le 10 \\ & x_1,\ x_2,\ x_3 \ge 0 \\ & 2.925x_1 + 2.275x_2 + 8.125x_3 - 100.000\zeta_1 \ge -111.250 \\ & -2.925x_1 - 6.500x_2 + 0.325x_3 - 130.000\zeta_1 \ge -159.250 \\ & 0.625 \le \zeta_1 \le 1.600 \\ & 4.821x_1 + 0.673x_2 + 4.933x_3 - 100.000\zeta_2 \ge -107.424 \\ & -0.673x_1 - 4.821x_2 + 3.364x_3 - 123.333\zeta_2 \ge -141.646 \\ & 0.682 \le \zeta_2 \le 1.467 \\ & 5.143x_1 + 6.171x_2 + 0.343x_3 - 100.000\zeta_3 \ge -105.714 \\ & -1.029x_1 + 2.057x_2 - 4.457x_3 - 120.000\zeta_3 \ge -133.714 \\ & 0.714 \le \zeta_3 \le 1.400\,. \end{array}$$

We solve this and obtain and upper bound on the value of the subproblem P(Δ):

$$\bar{\mathbf{x}} = (0.000,\ 0.325,\ 1.128)\,, \qquad \bar{z}(\Delta) = 3.422\,, \qquad \Sigma_{i=1}^{3}\ \bar{\eta}_i/\bar{\xi}_i = 3.064$$
$$(\bar{\boldsymbol{\xi}};\ \bar{\boldsymbol{\eta}}) = (55.011,\ 51.302,\ 55.163;\ 56.941,\ 53.232,\ 54.685)\,.$$

Since $\bar{z}(\Delta) - z^{\epsilon} = 0.407 \geq 10^{-3}$, we see that Δ needs dividing into Δ^2 and Δ^3. In addition, since $\Sigma_{i=1}^{p}\ \bar{\eta}_i/\bar{\xi}_i = 3.064 > z^{\epsilon}$, we update the incumbent:

$$\mathbf{x}^{\epsilon} = (0.000,\ 0.325,\ 1.128)\,, \qquad z^{\epsilon} = 3.064$$
$$(\boldsymbol{\xi}^{\epsilon};\ \boldsymbol{\eta}^{\epsilon}) = (55.011,\ 51.302,\ 55.163;\ 56.941,\ 53.232,\ 54.685)\,.$$

To apply the ω-division rule, we choose an index:

$$\begin{aligned} 1 \in \arg\max\{\phi_i(\bar{\xi}_i,\ \bar{\eta}_i) - \bar{\eta}_i/\bar{\xi}_i \mid i = 1, 2, 3\} \\ = \arg\max\{1.212 - 1.035,\ 1.132 - 1.038,\ 1.078 - 0.991\} \\ = \arg\max\{0.176,\ 0.094,\ 0.087\}\,, \end{aligned}$$

and let $w_1 = \bar{\eta}_1/\bar{\xi}_1 = 1.035$. Then we divide Δ_1 into

$$\Delta_1^2 = \{(\xi_1, \eta_1) \in \mathbb{R}_+^2 \mid 0.625\xi_1 \leq \eta_1 \leq 1.035\xi_1\}$$
$$\Delta_1^3 = \{(\xi_1, \eta_1) \in \mathbb{R}_+^2 \mid 1.035\xi_1 \leq \eta_1 \leq 1.600\xi_1\}\,,$$

and add $\Delta^3 = \Delta_1^3 \times \Delta_2 \times \Delta_3$ and $\Delta^2 = \Delta_1^2 \times \Delta_2 \times \Delta_3$ to the top of the stack $\mathcal{J}$ in this order.

Iteration 2. We take $\Delta = \Delta^2$ out of $\mathcal{J} = \{2, 3\}$. Since Δ_2 and Δ_3 remain the same as before, only the overestimator of ξ_1/η_1 needs changing:

$$\phi_1(\xi_1, \eta_1) = \min\left\{\begin{matrix} -0.013\xi_1 + 0.020\eta_1 + 0.625 \\ -0.013\xi_1 + 0.013\eta_1 + 1.035 \end{matrix}\right\}.$$

As a result, the linear program equivalent to $\bar{\mathrm{P}}(\Delta)$ is different in only three constraints from the previous one. By solving it, we see that $\bar{\mathbf{x}}$ and $(\bar{\boldsymbol{\xi}},\ \bar{\boldsymbol{\eta}})$ remains optimal but $\bar{z}(\Delta)$ changes:

$$\bar{z}(\Delta) = 3.245\,, \qquad \sum_{i=1}^{3}\ \bar{\eta}_i/\bar{\xi}_i = 3.064\,.$$

Since $\bar{z}(\Delta) - z^{\epsilon} = 0.181 > 10^{-3}$, we divide Δ further. Choosing

$$\begin{aligned} 2 \in \arg\max\{\phi_i,\ \bar{\eta}_i) - \bar{\eta}_i/\bar{\xi}_i \mid i = 1, 2, 3\} \\ = \arg\max\{0.000,\ 0.094,\ 0.087\} \end{aligned}$$

and letting $w_2 = \bar{\eta}_2/\bar{\xi}_2 = 1.038$, we have

$$\Delta_2^4 = \{(\xi_2, \eta_2) \in \mathbb{R}_+^2 \mid 0.682\xi_2 \leq \eta_2 \leq 1.038\xi_2\}$$
$$\Delta_2^5 = \{(\xi_2, \eta_2) \in \mathbb{R}_+^2 \mid 1.038\xi_2 \leq \eta_2 \leq 1.467\xi_2\}\,,$$

and add $\Delta^5 = \Delta_1 \times \Delta_2^5 \times \Delta_3$ and $\Delta^4 = \Delta_1 \times \Delta_2^4 \times \Delta_3$ to $\mathcal{J}$.

Iterations 3, 4. As in the previous iterations, we take $\Delta = \Delta^4$ out of $\mathcal{J} = \{4, 5, 3\}$ and obtain

$$\bar{z}(\Delta) = 3.151\,, \qquad \sum_{i=1}^{3} \bar{\eta}_i / \bar{\xi}_i = 3.063\,.$$

Since $\bar{z}(\Delta) - z^\epsilon = 0.087 > 10^{-3}$, we choose Δ_3 and divide it into

$$\Delta_3^6 = \{(\xi_3, \eta_3) \in \mathbb{R}_+^2 \mid 0.714\xi_3 \le \eta_3 \le 0.990\xi_3\}$$
$$\Delta_3^7 = \{(\xi_3, \eta_3) \in \mathbb{R}_+^2 \mid 0.990\xi_3 \le \eta_3 \le 1.400\xi_3\}\,.$$

We next take $\Delta = \Delta^6$ out of $\mathcal{J} = \{6, 7, 5, 3\}$ and obtain

$$\bar{z}(\Delta) = 3.063\,, \qquad \sum_{i=1}^{3} \bar{\eta}_i / \bar{\xi}_i = 3.063\,.$$

Since $\bar{z}(\Delta) - z^\epsilon = -0.001 < 10^{-3}$, we see that Δ contains no optimal solutions to (5.1); and we backtrack to the top of $\mathcal{T} = \{7, 5, 3\}$.

After 21 iterations, we have $\mathcal{J} = \emptyset$ and a 10^{-3}-optimal solution to (5.1):

$$\mathbf{x}^\epsilon = (0.000,\ 0.000,\ 1.250)\,, \qquad z^\epsilon = 3.074$$
$$(\boldsymbol{\xi}^\epsilon;\ \boldsymbol{\eta}^\epsilon) = (53.750,\ 50.000,\ 55.000;\ 56.250,\ 52.500,\ 53.750)\,.$$

The total numbers of iterations and branching operations are 25 and 13, respectively; and the incumbent is updated six times. If we employ the bisection rule instead of ω-division, the incumbent is updated five times while the total numbers of iterations and branching operations are 103 and 52, respectively.

For the readers' information, we give the result on another instance slightly larger than (5.1):

$$\begin{array}{ll} \text{maximize} & z = \dfrac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50} + \dfrac{3x_1 + 4x_3 + 50}{4x_1 + 4x_2 + 5x_3 + 50} + \\ & \dfrac{x_1 + 2x_2 + 5x_3 + 50}{x_1 + 5x_2 + 5x_3 + 50} + \dfrac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50} \\ \text{subject to} & 2x_1 + x_2 + 5x_3 \le 10,\ x_1 + 6x_2 + 3x_3 \le 10 \\ & 5x_1 + 9x_2 + 2x_3 \le 10,\ 9x_1 + 7x_2 + 3x_2 \le 10 \\ & x_1, x_2, x_3 \ge 0\,. \end{array} \tag{5.2}$$

If we employ the depth-first and ω-division rule, we have a 10^{-3}-optimal solution to (5.2) in 31 iterations and 15 branching operations:

$$\mathbf{x}^\epsilon = (0.000,\ 1.111,\ 0.000)\,, \qquad z^\epsilon = 4.217$$
$$(\boldsymbol{\xi}^\epsilon;\ \boldsymbol{\eta}^\epsilon) = (53.333,\ 50.000,\ 52.222,\ 52.222;\ 53.333,\ 54.444,\ 55{,}556,\ 55.556)\,.$$

The incumbent is updated three times. When we employ the bisection rule, it is updated five times while the total numbers of iterations and branching operations are 117 and 59, respectively.

6. Experiment with the algorithm

In this section, we will report computational results of testing the algorithm SUMRATIO on randomly generated problems, which were of the form:

$$\begin{array}{ll} \text{maximize} & z = \displaystyle\sum_{i=1}^{p} \frac{\displaystyle\sum_{j=1}^{n'} d_{ij}x_{ij} + c}{\displaystyle\sum_{j=1}^{n'} c_{ij}x_{ij} + c} \\ \text{subject to} & \displaystyle\sum_{j=1}^{n'} a_{kj}x_j \le 1.0,\ k = 1, \ldots, m \\ & x_j \ge 0.0,\ j = 1, \ldots, n' . \end{array} \tag{6.1}$$

Data c_{ij}, $d_{ij} \in [0.0, 0.5]$ and $a_{kj} \in [0.0, 1.0]$ were uniformly random numbers. All constant terms of denominators and numerators were the same number c, which ranged between 2.0 and 100.0.

The algorithm was coded in double precision C language according to the description in Section 5. The tolerance ϵ was fixed at 10^{-5}. The numbers s_i^1, t_i^1, u_i and v_i were determined in the same way as in the case of example (5.1). In Step 1, depth first was adopted as the rule for selecting j from $\mathcal{J}$ in order to save on memory. In Step 2, the linear programming problem (3.5) was solved to compute $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$ and $\bar{z}(\Delta)$. Starting from the preceding solution, we restored the primal feasibility of (3.5) by applying some dual simplex pivoting operations. As the rule for dividing Δ in Step 3, we tried both bisection and ω-division. The code adopting the former was named SR_2 and the latter SR_O. Both were tested on a Unix workstation (UltraSPARC-IIi, 440 MHz).

6.1. COMPUTATIONAL RESULTS

Figure 3 depicts the average performance of the algorithm SUMRATIO on ten instances of size $(m, n') = (60, 40)$ for each p when the value of c was fixed at 10.0. The size of p was made to change by 2 each from 2 to 12. We see from the line graph at the top that SR_O requires more branching operations that SR_2 for p greater than 7. This is an unexpected result in comparison with the usual rectangular branch-and-bound method for separable concave minimization problems (see e.g., Remark 5.6 in [21]). As is shown by the graph at the bottom, however, SR_O requires less CPU time than SR_2 for all p up to 10. In the ω-division rule, $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}})$ always belongs to both Δ^{2k} and Δ^{2k+1}. Therefore, the feasibility of (3.5) can recover

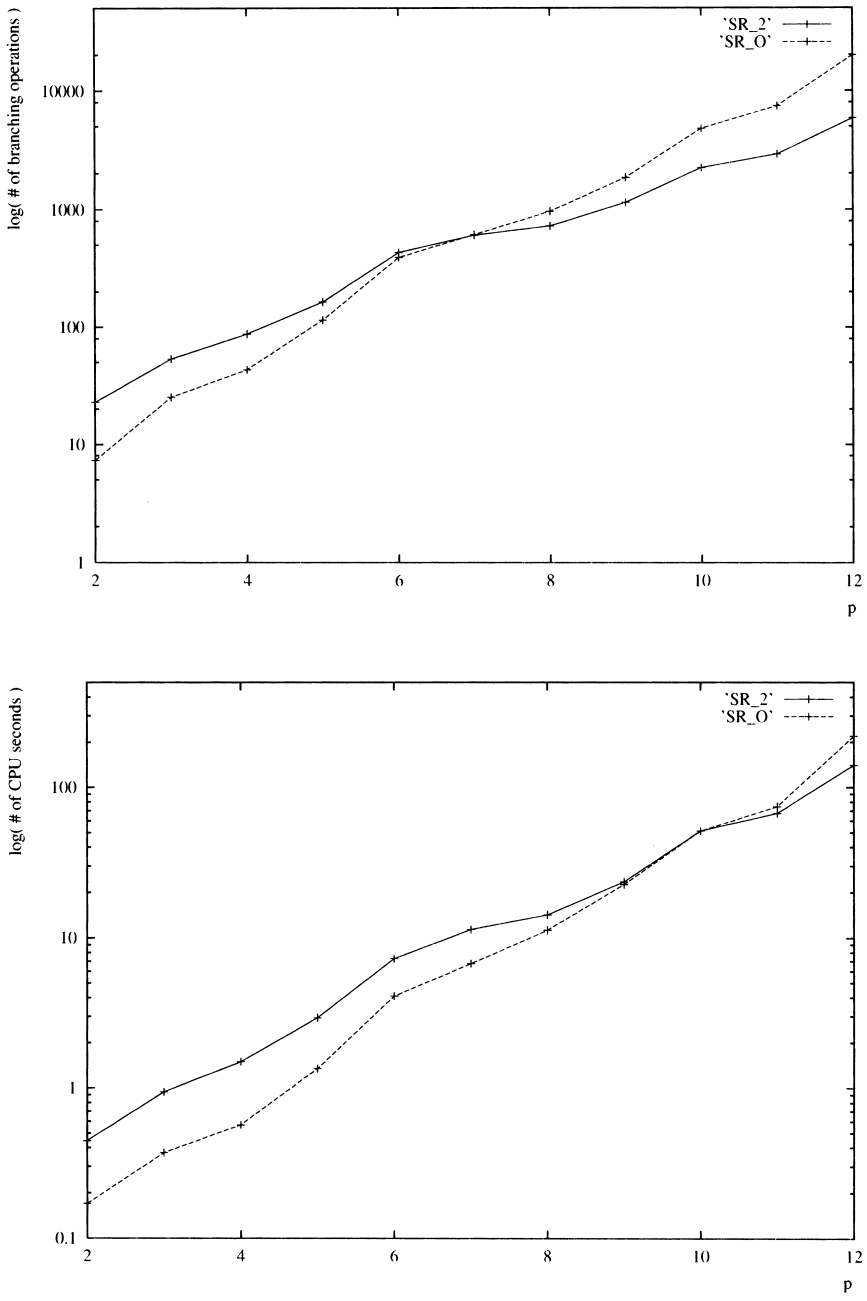

Figure 3. Behavior of SUMRATIO when $(m, n') = (60, 40)$ and $c = 10.0$.

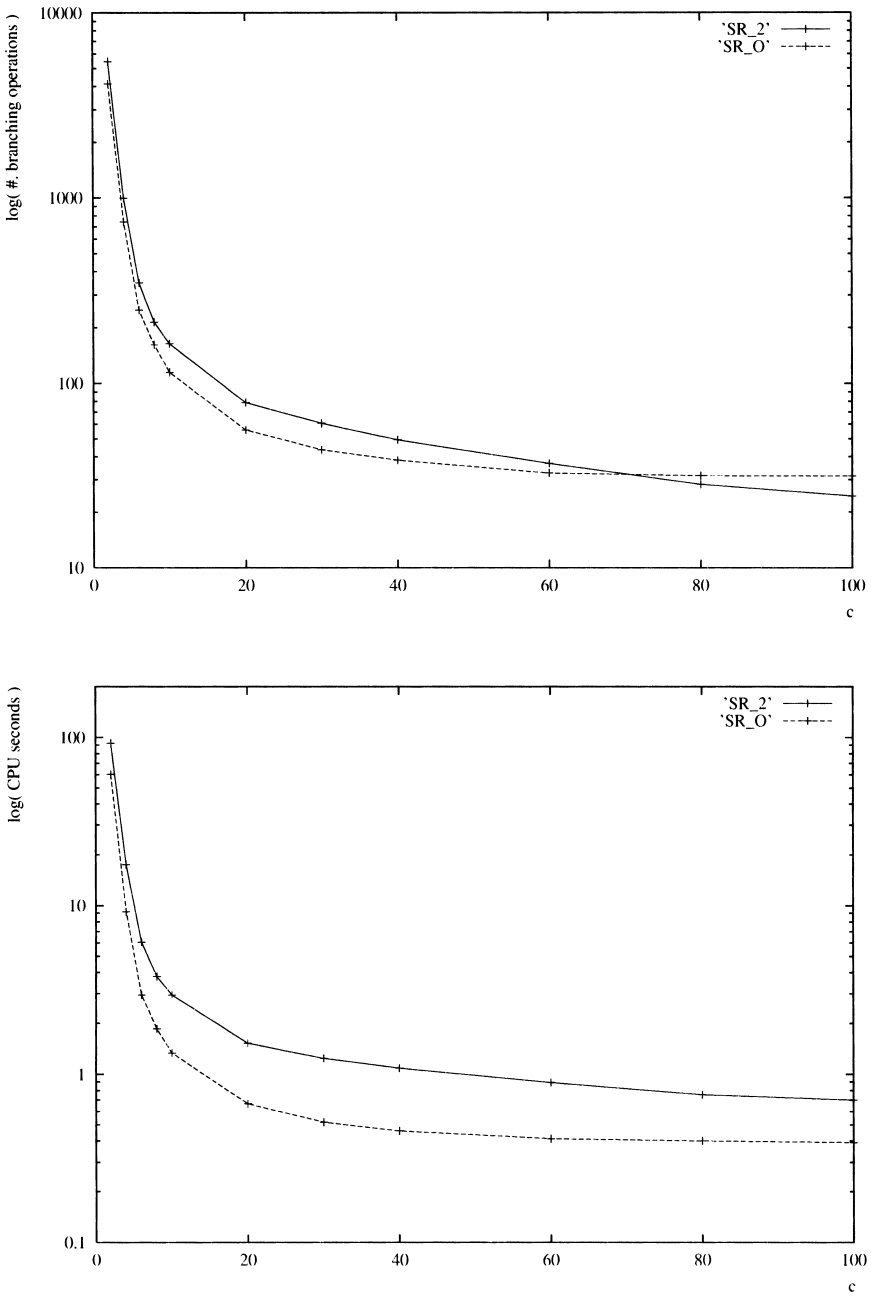

Figure 4. Behavior of SUMRATIO when $(m, n', p) = (60, 40, 5)$.

Table 1. Computational results of SR_O when $c = 10.0$

$m \times n'$	$p = 3$		$p = 4$		$p = 5$		$p = 6$	
	Branch	Time	Branch	Time	Branch	Time	Branch	Time
40×60	24.2 (12.9)	0.605 (0.308)	43.5 (12.64)	0.568 (0.125)	114 (64.73)	1.35 (0.797)	390 (335)	4.10 (3.98)
80×60	22.5 (7.15)	1.03 (0.231)	108 (94.5)	3.22 (3.24)	237 (182)	6.85 (6.10)	647 (417)	18.3 (15.3)
60×80	34.5 (11.6)	1.62 (0.460)	117 (107)	4.85 (4.04)	304 (200)	11.0 (6.47)	613 (459)	20.9 (17.2)
100×80	27.7 (14.1)	2.35 (0.627)	107 (75.5)	5.51 (3.14)	412 (480)	17.9 (21.4)	521 (545)	22.3 (23.4)
80×100	57.1 (31.6)	4.59 (2.15)	138 (114)	9.35 (6.15)	365 (299)	22.8 (18.5)	989 (1,385)	66.7 (99.3)
120×100	51.2 (28.7)	6.70 (2.93)	83.9 (55.2)	8.86 (3.45)	500 (537)	40.7 (41.2)	1,045 (1,045)	114 (169)

quickly whichever cone is chosen from $\mathcal{J}$ in the next iteration. In other words, the less CPU time of SR_O is due to its fewer simplex pivoting operations.

Figure 4 gives the results on instances of size $(m, n', p) = (60, 40, 5)$ for 11 different values of c from 2.0 to 100.0 These two line graphs show that the algorithm SUMRATIO is very sensitive to the magnitude of c, whether it uses bisection or ω-division. For a small c, each trapezoid $\Gamma_i \cap \Delta_i$ is defined near the origin in the ξ_i-η_i plane. In that case, η_i/ξ_i is quite different from linear in shape; and hence ϕ_i defined only by two affine functions is too simple to estimate η_i/ξ_i precisely. In contrast to this, ϕ_i can make a fine estimate of η_i/ξ_i if $\Gamma_i \cap \Delta_i$ is far away from the origin, i.e., c is a large number. We can recognize from the figure that such a fine estimate is given when c is greater than 20.0.

Based upon the above observations, we tried to solve larger-size problems with c fixed at 10.0 using the ω-division code SR_O. The results are listed in Table 1. The columns labeled 'Branch' and 'Time' contain the average number of branching operations and the average CPU time in seconds, respectively, required to solve ten instances of size up to $(m, n', p) = (120, 100, 6)$. The figures in parentheses are their standard deviations. We see from this table that SR_O is rather insensitive to the size (m, n') and can solve fairly large-size problems as long as p is less than 7. Since we have not yet compared our algorithm with other existing ones, we cannot make a final conclusion. At least for the randomly generated class (6.1), however, these computational results will support our claim that SUMRATIO can serve as a practical deterministic algorithm.

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